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ON GLOBAL VOLUME CHANGES ACCOMPANYING CERTAIN PLANE-STRAIN NONLINEAR ELASTIC DEFORMATIONS

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Abstract—It is shown in this paper that for certain classes of unconstrained isotropic elastic solids, predictions can be made as to whether the overall volume changes which accompany two nonlinear elastic deformations (namely the radial deformation of cylindrical tubes and the bending of rectangular blocks into annular cylindrical sectors) are expansive or contractive. It is emphasised that such predictions may be viewed as universal relations for the material classes concerned. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

One of the issues of long-standing concern to material scientists has been the measurement of volume changes that accompany the deformations of elastic solids. Discussions relating to comparison between theory and experiments, and relevant references, can be found in the book by Treloar (1975) and in the review papers by Price (1976) and Ogden (1982). More recently, global volume changes data for cylindrical samples subjected to torsional deformations have been obtained by Pixa *et al.* (1988) and Duran and McKenna (1990).

As it is well known, when an elastic body is subjected to boundary conditions of displacement type the final shape of the body, and consequently its final volume, is specified at the outset. This is no longer the case, however, when such a body is subjected to boundary conditions of a different nature and in these circumstances, for the same boundary data, the same type of overall deformation (such as the radial deformation of circular cylindrical tubes subjected to prescribed internal pressure) could be accompanied either by an overall volume increase, or by an overall volume decrease, according to the kind of material of which the elastic body is composed.

Under the assumption that the body forces are zero, we show in this paper that, for certain material classes, non-trivial predictions can be made as to whether the overall volume changes that accompany two types of overall deformation are expansive or contractive. These deformations are (i) the radial deformation of circular cylindrical tubes and (ii) the bending of rectangular blocks into annular cylindrical sectors and we note that such predictions may be viewed as universal relations for the material classes concerned. (Indeed, if in an experiment it is found that a certain overall deformation is accompanied by an overall volume increase (decrease), it can be concluded that the material involved in the experiment cannot belong to the class of materials for which it is known that this deformation is necessarily accompanied by an overall volume decrease (increase); see Beatty (1987) for a discussion on the role of universal relations in Continuum Mechanics.) The tools which we employ for our analysis are the Jensen's inequality (Hardy *et al.*, 1952, p. 150) and certain inequalities for symmetric functions of two variables which are consequences of a result of Aron (1991). The materials under consideration include well-known material models for solid and foam rubbers and we note that some other nonlinear elastic deformations (such as the bending of an annular cylindrical sector into a similar sector) are also amenable to treatment by the present method of investigation.

2. PRELIMINARIES

In this paper we are concerned with plane deformations $\mathbf{x} = \mathbf{x}(\mathbf{X})$ where $\mathbf{x}$ and $\mathbf{X}$ are points that belong, respectively, to the domains $\bar{\Omega}$ and Ω of the Euclidean space R^2. The deformation gradient $\mathbf{F} \equiv \nabla\mathbf{x}$ is assumed to satisfy the condition $\det \mathbf{F} > 0$, where det stands for determinant. By the polar decomposition theorem the deformation gradient can be represented uniquely in the form $\mathbf{F} = \mathbf{VR}$ where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{V}$ is a symmetric positive-definite tensor. The eigenvalues of $\mathbf{V}$, denoted here by λ_i, $i = 1, 2$, are known as the principal stretches.

For plane deformations, the strain-energy function W, which characterises a homogeneous isotropic hyperelastic solid, is a symmetric function of λ_1, λ_2:

$$W = W(\lambda_1, \lambda_2) = W(\lambda_2, \lambda_1). \tag{1}$$

The function W will be assumed to satisfy

$$W(\lambda_1, \lambda_2) \geqslant 0, \quad W_i(1,1) = 0, \quad W_i \equiv \frac{\partial W}{\partial \lambda_i}, \quad i = 1, 2, \tag{2}$$

and

$$(\lambda - 1) W_1(\lambda, \lambda) > 0, \quad \text{for all } \lambda > 0, \tag{3}$$

equality in $(2)_1$ being possible if and only if $\lambda_1 = \lambda_2 = 1$. The condition (3) is the well-known pressure–compression inequality (which is the requirement that, for deformations with $\lambda_1 = \lambda_2 = \lambda$, the volume is decreased by pressure but increased by tension; see Truesdell and Noll, 1965, Section 51) and the condition $(2)_2$ expresses the fact that the reference configuration is unstressed.

According to a result of Aron (1991), a symmetric function of two variables, $f = f(\lambda_1, \lambda_2)$, satisfies

$$f(\lambda_1, \lambda_2) > f(\sqrt{\lambda_1 \lambda_2}, \sqrt{\lambda_1 \lambda_2}), \quad \lambda_1 \neq \lambda_2, \tag{4}$$

provided that the following two conditions are satisfied:

$$\lambda_1 f_1 - \lambda_2 f_2 \neq 0, \quad \text{for } \lambda_1 \neq \lambda_2, \tag{5}$$

and

$$E_f(\lambda) \equiv f_{11}(\lambda, \lambda) - f_{12}(\lambda, \lambda) + \frac{1}{\lambda} f_1(\lambda, \lambda) > 0, \quad \text{for all } \lambda > 0, \tag{6}$$

where

$$f_{ij} \equiv \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}, \quad i, j = 1, 2, 3. \tag{7}$$

A trivial modification of the proof of (4), shows that the conditions $E_f(\lambda) < 0$ (for all $\lambda > 0$) and (5) (together) imply

$$f(\lambda_1, \lambda_2) < f(\sqrt{\lambda_1 \lambda_2}, \sqrt{\lambda_1 \lambda_2}), \quad \lambda_1 \neq \lambda_2. \tag{8}$$

Using these results, it was shown by Aron and Aizicovici (1994) that for elastic materials which satisfy

$$\lambda_1 W_1 - \lambda_2 W_2 + \lambda_1^2 W_{11} - \lambda_2^2 W_{22} \neq 0, \quad \text{for } \lambda_1 \neq \lambda_2, \tag{9}$$

we have

$$E_p(\lambda) > 0(\text{for all } \lambda > 0) \Rightarrow p(\lambda_1, \lambda_2) > p(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}), \quad \lambda_1 \neq \lambda_2, \tag{10}$$

and

$$E_p(\lambda) < 0(\text{for all } \lambda > 0) \Rightarrow p(\lambda_1, \lambda_2) < p(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}), \quad \lambda_1 \neq \lambda_2, \tag{11}$$

where we have used the notations

$$E_p(\lambda) \equiv 3W_{11}(\lambda,\lambda) - W_{12}(\lambda,\lambda) + \lambda\left[W_{111}(\lambda,\lambda) - W_{112}(\lambda,\lambda) + \frac{1}{\lambda^2}W_1(\lambda,\lambda)\right], \tag{12}$$

and

$$p(\lambda_1, \lambda_2) \equiv \lambda_1 W_1(\lambda_1, \lambda_2) + \lambda_2 W_2(\lambda_1, \lambda_2). \tag{13}$$

Some examples of strain-energy functions which satisfy $(10)_2$ and $(11)_2$ can be found in Aron and Aizicovici (1994) and we note that, from (9) and (10), it follows that a strain-energy function of the form

$$W = g(\lambda_1^\alpha + \lambda_2^\alpha) + h(J), \quad J \equiv \lambda_1\lambda_2, \quad \alpha = \text{const.}, \quad \alpha \neq 0, \tag{14}$$

and which is such that

$$\alpha^3[g'(x) + xg''(x)] > 0, \quad \text{for all } x > 0, \tag{15}$$

satisfies $(10)_{2,3}$. Also, from (9) and (11), it follows that a strain-energy function of the form (14) and which is such that

$$\alpha^3[g'(x) + xg''(x)] < 0, \quad \text{for all } x > 0, \tag{16}$$

satisfies $(11)_{2,3}$. The physical significance of inequalities $(10)_{2,3}$ and $(11)_{2,3}$ has been discussed by Aron and Aizicovici (1994) in the context of both homogeneous and non-homogeneous deformations.

Based upon the result of Aron (1991) mentioned above, we can also show that for elastic materials which satisfy the condition

$$\lambda_1^2 W_{11} - \lambda_2^2 W_{22} \neq 0, \quad \text{for } \lambda_1 \neq \lambda_2, \tag{17}$$

we have

$$E_q(\lambda) > 0(\text{for all } \lambda > 0) \Rightarrow q(\lambda_1, \lambda_2) > q(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}), \quad \lambda_1 \neq \lambda_2, \tag{18}$$

and

$$E_q(\lambda) < 0(\text{for all } \lambda > 0) \Rightarrow q(\lambda_1, \lambda_2) < q(\sqrt{\lambda_1\lambda_2}, \sqrt{\lambda_1\lambda_2}), \quad \lambda_1 \neq \lambda_2, \tag{19}$$

where

$$E_q(\lambda) \equiv -2W_{11}(\lambda,\lambda)+\lambda[W_{112}(\lambda,\lambda)-W_{111}(\lambda,\lambda)], \tag{20}$$

and

$$q(\lambda_1,\lambda_2) \equiv W(\lambda_1,\lambda_2)-\lambda_1 W_1(\lambda_1,\lambda_2)-\lambda_2 W_2(\lambda_1,\lambda_2). \tag{21}$$

Using (17) and (18), it can be easily checked that a strain-energy function of the form (14), and which is such that

$$(\alpha-1)g'(x)+xg''(x) < 0, \quad \text{for all } x > 0, \tag{22}$$

satisfies $(18)_{2,3}$. Similarly, it follows from (17) and (19) that a strain-energy function of the form (14), and which is such that

$$(\alpha-1)g'(x)+xg''(x) > 0, \quad \text{for all } x > 0, \tag{23}$$

satisfies $(19)_{2,3}$.

Examples of strain energies that satisfy $(19)_{2,3}$ but which, in general, do not satisfy either $(10)_{2,3}$ or $(11)_{2,3}$, are provided by the harmonic materials introduced by John (1960), which are characterised by a strain-energy function of the form

$$W = g(\lambda_1+\lambda_2)-J, \quad g(2) = g'(2) = 1, \quad g'' > 0. \tag{24}$$

A specific example, which illustrates this particular point, is provided by the semi-linear material (see John, 1960) for which

$$\left.\begin{aligned} g(\lambda_1+\lambda_2) &= \frac{\lambda_0+2\mu_0}{4\mu_0}(\lambda_1+\lambda_2-2)^2+\lambda_1+\lambda_2-1 \\ \lambda_0,\mu_0 &= \text{const.}, \quad \mu_0 > 0, \quad \lambda_0+\mu_0 > 0. \end{aligned}\right\} \tag{25}$$

On the other hand, it is clear that the compressible Varga materials (which have been introduced by Haughton (1987) and independently by Carroll (1988), and which are obtained by letting $\alpha = 1$ and $g(x) = x$ in (14)) cannot satisfy either $(18)_{2,3}$ or $(19)_{2,3}$, although they always satisfy $(10)_{2,3}$. The compressible Varga strain-energy function, however, satisfies the condition

$$q(\lambda_1,\lambda_2) = q(\sqrt{\lambda_1\lambda_2},\sqrt{\lambda_1\lambda_2}), \quad \text{for all } \lambda_1,\lambda_2 > 0, \tag{26}$$

and we note that a more general strain-energy function which belongs to the class (1) and for which (26) holds is

$$W = \sqrt{J}V\left(\frac{\lambda_1}{\lambda_2}\right)+h(J), \tag{27}$$

where V is a symmetric function of λ_1, λ_2.

Examples of a more specific nature will be considered in Section 5 of this paper and we note that in what follows, the above results will be used in conjunction with Jensen's inequality (Hardy *et al.*, 1952, p. 150)

$$\iint_\Omega \phi(\det \mathbf{F})\, \mathrm{d}\Omega \geqslant A(\Omega)\phi\left(A(\Omega)^{-1}\iint_\Omega (\det \mathbf{F})\, \mathrm{d}\Omega\right), \tag{28}$$

where ϕ denotes a convex function of class C^2 and $A(\Omega)$ stands for the area of Ω.

3. VOLUME CHANGES ACCOMPANYING THE BENDING OF RECTANGULAR BLOCKS

Let (r, θ) denote spatial polar coordinates and (X, Y) referential Cartesian coordinates with

$$-D \leqslant X \leqslant D, \quad -\tfrac{1}{2} \leqslant Y \leqslant \tfrac{1}{2}, \quad D = \text{const.} \tag{29}$$

A deformation which describes the bending of a rectangular block into a sector of a circular tube may be given in the form

$$r = f(X), \quad \theta = \alpha Y, \tag{30}$$

where the function f and the constant α are to be determined from the boundary conditions and the equilibrium equation.

For deformations (30) and materials (1), the equilibrium equations in the absence of body forces can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}X} W_1(\lambda_1, \lambda_2) = \alpha W_2(\lambda_1, \lambda_2), \tag{31}$$

where

$$\lambda_1 = f'(X), \quad \lambda_2 = \alpha f(X), \tag{32}$$

and the prime denotes the derivative with respect to the argument (see Ogden, 1984a, Section 5.2.4).

As discussed by Ogden (1984a, Section 5.2.4), we may consider the equation (31) together with one of the following two sets of boundary conditions:

$$\left.\begin{array}{l} W_1(f'(D), \alpha f(D)) = W_1(f'(-D), \alpha f(-D)) = 0, \\ \alpha \text{ prescribed}, \end{array}\right\} \tag{33}$$

or

$$\left.\begin{array}{l} W_1(f'(D), \alpha f(D)) = W_1(f'(-D), \alpha f(-D)) = 0, \\ M = \displaystyle\int_{-D}^{D} f(X) W_2(f'(X), \alpha f(X))\,\mathrm{d}X \quad \text{prescribed}. \end{array}\right\} \tag{34}$$

The equations $(33)_{1,2}$ (or $34)_{1,2}$) imply that the tractions on both curved boundaries of the (deformed) body vanish and we note that the expression in the right-hand side of $(34)_3$ represents the magnitude (up to a multiplicative constant which depends upon the dimensions of the rectangular block) of the moment about the origin of the stresses on the faces $\theta = \pm\alpha/2$.

Let Ω denote the domain defined by (29) and let (f, α) be a solution to any of the two considered boundary value problems, which we now assume exists. On using (31), (32) and $(33)_{1,2}$, and on integrating by parts, we find that (see (11))

$$\iint_{\Omega} p(f', \alpha f)\,\mathrm{d}\Omega = \int_{-D}^{D} [f' W_1(f', \alpha f) + \alpha f W_2(f', \alpha f)]\,\mathrm{d}X = 0. \tag{35}$$

Thus, if W is a material which satisfies $(10)_{2,3}$, we infer from (35) that we have

$$\iint_{\Omega} W_1(\sqrt{\alpha ff'}, \sqrt{\alpha ff'})\, d\Omega \leqslant 0. \tag{36}$$

Introducing the additional assumption

$$\frac{d^2}{d\lambda^2} W_1(\sqrt{\lambda}, \sqrt{\lambda}) \geqslant 0, \quad \text{for all } \lambda > 0, \tag{37}$$

and making use of (28) now leads to

$$W_1(\sqrt{\delta}, \sqrt{\delta}) \leqslant 0, \tag{38}$$

where we have used the notation

$$\delta \equiv A(\Omega)^{-1} \iint_{\Omega} (\det \mathbf{F})\, d\Omega = (2D)^{-1} \int_{-D}^{D} \alpha ff'\, dX = \alpha(4D)^{-1}[f^2(D) - f^2(-D)]. \tag{39}$$

In view of (3), the inequality (38) yields

$$\delta \leqslant 1 \tag{40}$$

and, since the integral in $(39)_1$ represents the deformed area, we conclude that under our constitutive assumptions the deformation cannot be accompanied by an overall volume increase.

In a similar manner, we can show that if the material satisfies $(11)_{2,3}$ and

$$\frac{d^2}{d\lambda^2} W_1(\sqrt{\lambda}, \sqrt{\lambda}) \leqslant 0, \quad \text{for all } \lambda > 0, \tag{41}$$

we must necessarily have

$$\delta \geqslant 1, \tag{42}$$

so that, in this case, the deformation cannot be accompanied by an overall volume decrease.

We assume now that the elastic material satisfies $(19)_{2,3}$ and, additionally,

$$\frac{d}{d\lambda} q(\sqrt{\lambda}, \sqrt{\lambda}) = \frac{d}{d\lambda}[W(\sqrt{\lambda}, \sqrt{\lambda}) - 2\sqrt{\lambda} W_1(\sqrt{\lambda}, \sqrt{\lambda})] < 0, \quad \text{for all } \lambda > 0, \tag{43}$$

and

$$\frac{d^2}{d\lambda^2} q(\sqrt{\lambda}, \sqrt{\lambda}) \leqslant 0, \quad \text{for all } \lambda > 0. \tag{44}$$

We note that the inequality (43), which can be written in the form

$$2\sqrt{\lambda} \frac{d}{d\lambda} W_1(\sqrt{\lambda}, \sqrt{\lambda}) = W_{11}(\sqrt{\lambda}, \sqrt{\lambda}) + W_{12}(\sqrt{\lambda}, \sqrt{\lambda}) > 0, \quad \text{for all } \lambda > 0, \tag{45}$$

is a well-known (necessary) condition for the stability of equibiaxial deformations under prescribed dead-loading boundary tractions (see Ogden, 1984b for instance) and that it implies the pressure–compression condition (3).

From $(19)_{2,3}$ and $(2)_1$ (or, alternatively, from (26) and $(2)_1$), we obtain

$$-\lambda_1 W_1(\lambda_1,\lambda_2)-\lambda_2 W_2(\lambda_1,\lambda_2) < W(\sqrt{\lambda_1\lambda_2},\sqrt{\lambda_1\lambda_2})-2\sqrt{\lambda_1\lambda_2}W_1(\sqrt{\lambda_1\lambda_2},\sqrt{\lambda_1\lambda_2}),$$
$$(\lambda_1,\lambda_2)\neq(1,1), \quad (46)$$

which, when combined with (35), leads to

$$\iint_\Omega [W(\sqrt{\alpha ff'},\sqrt{\alpha ff'})-2\sqrt{\alpha ff'}W_1(\sqrt{\alpha ff'},\sqrt{\alpha ff'})]\,\mathrm{d}\Omega \geqslant 0. \quad (47)$$

Use of (44) with (28) now gives

$$W(\sqrt{\delta},\sqrt{\delta})-2\sqrt{\delta}W_1(\sqrt{\delta},\sqrt{\delta}) \geqslant 0 \quad (48)$$

and, in view of (43) and the fact that $q(1,1) = 0$, we infer that (48) implies (40). Thus, for materials which satisfy $(19)_{2,3}$, (43) and (44) (or (26), (43) and (44)), the considered deformation cannot be accompanied by an overall volume increase.

As discussed in the Introduction, the inequalities (40) and (42) may be regarded as universal relations. Thus, if under appropriately simulated experimental conditions, it is found that the bending of a rectangular block into an annular cylindrical sector is accompanied by an overall volume increase, it can be concluded that the material involved in the experiment cannot belong to the class of isotropic elastic solids which satisfy (9), $(10)_1$ and (37), or to the class of isotropic elastic solids that satisfy (17), $(19)_1$, (43) and (44), or to the class of isotropic elastic solids that satisfy (26), (43) and (44). On the other hand, if it is found experimentally that the considered deformation is accompanied by an overall volume decrease, it can be concluded that the material involved in the experiment cannot belong to the class of isotropic elastic solids which satisfy (9), $(11)_1$ and (41).

4. VOLUME CHANGES ACCOMPANYING THE RADIAL DEFORMATIONS OF CYLINDRICAL SHELLS

Let (r,θ) denote spatial polar coordinates and (R,Θ) referential polar coordinates with

$$0<\rho_1<R<\rho_2, \quad 0<\Theta\leqslant 2\pi, \quad \rho_1,\rho_2=\text{const.} \quad (49)$$

The radial deformations of cylindrical shells are characterised by

$$r=f(R), \quad \theta=\Theta, \quad (50)$$

where the function f is to be determined from the equilibrium equation and the boundary conditions.

For deformations (50) and materials (1), the equilibrium condition in the absence of body forces is (see Ogden, 1984a; Section 5.2.4)

$$\frac{\mathrm{d}}{\mathrm{d}R}[RW_1(\lambda_1,\lambda_2)] = W_2(\lambda_1,\lambda_2), \quad (51)$$

where

$$\lambda_1=f'(R), \quad \lambda_2=\frac{f(R)}{R}. \quad (52)$$

On using (51) and (52) and on integrating over Ω (where Ω is now the domain characterised by (49)) we obtain (see (13))

$$\iint_{\Omega} p\left(f', \frac{f}{R}\right) \mathrm{d}\Omega = 2\pi[\rho_2 f(\rho_2) W_1|_{R=\rho_2} - \rho_1 f(\rho_1) W_1|_{R=\rho_1}] \tag{53}$$

which, when combined with $(10)_{2,3}$, gives

$$\pi[\rho_2 f(\rho_2) W_1|_{R=\rho_2} - \rho_1 f(\rho_1) W_1|_{R=\rho_1}] \geqslant \iint_{\Omega} \sqrt{\frac{ff'}{R}} W_1\left(\sqrt{\frac{ff'}{R}}, \sqrt{\frac{ff'}{R}}\right) \mathrm{d}\Omega. \tag{54}$$

With the help of (28) we now find that (54), together with the additional assumption

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}[\sqrt{\lambda} W_1(\sqrt{\lambda}, \sqrt{\lambda})] \geqslant 0, \quad \text{for all } \lambda > 0, \tag{55}$$

yields

$$\pi[\rho_2 f(\rho_2) W_1|_{R=\rho_2} - \rho_1 f(\rho_1) W_1|_{R=\rho_1}] \geqslant A(\Omega)\sqrt{\delta} W_1(\sqrt{\delta}, \sqrt{\delta}). \tag{56}$$

Thus, in view of (3), we conclude that (40) (where δ now denotes the ratio of the cross-sectional area of the cylindrical body in the deformed configuration to the cross-sectional area of the cylindrical body in the reference configuration) holds provided that the boundary conditions are chosen so as to ensure that the left-hand side of (56) is non-positive and provided that the material satisfies $(10)_{2,3}$ and (55). One possible set of boundary conditions is

$$W_1|_{R=\rho_1} = 0, \quad \left.\frac{W_1}{\lambda_2}\right|_{R=\rho_2} = -P_2, \quad P_2 = \text{const.}, P_2 > 0, \tag{57}$$

and clearly, there are some other such choices available.

By a similar argument, we can show that for the same choice of boundary conditions, (40) holds for all materials that satisfy (43), (44) and either one of $(19)_{2,3}$ and (26), whereas for boundary conditions which ensure that the left-hand side of (56) is non-negative, we can show that (42) holds, provided that the material satisfies $(11)_{2,3}$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}[\sqrt{\lambda} W_1(\sqrt{\lambda}, \sqrt{\lambda})] \leqslant 0, \quad \text{for all } \lambda > 0. \tag{58}$$

Clearly, in the present context and for the materials concerned, the eqns (40) and (42) may also be viewed as universal relations, along the lines discussed in the preceding section.

5. EXAMPLES

Our first example is provided by a material whose strain-energy function in plane strain is

$$W = \frac{\mu}{\beta}[\lambda_1^{-\beta} + \lambda_2^{-\beta} - 2 + \beta(J-1)], \quad \mu, \beta = \text{const.}, \mu, \beta > 0. \tag{59}$$

The material (59), which for $\beta = 2$ represents the well-known Blatz–Ko material model for foam rubbers, was considered recently by Silling (1991), Haughton (1990), Biwa (1995) and Wang and Aron (1996). This material belongs to the class (14) and satisfies (16). Thus, the condition $(11)_{2,3}$ is satisfied and it can be easily verified that the inequalities (41) and (58) are also satisfied. In view of the previous discussion we conclude that, for appropriate boundary conditions (as stated in Sections 3 and 4), the deformations (30) and (50) of bodies composed of this material cannot be accompanied by an overall volume decrease.

The compressible neo-Hookean material whose strain-energy function in plane strain is given by

$$W = \frac{\mu}{2}(\lambda_1^2+\lambda_2^2-3J^{2/3}+1)+\frac{k}{m}\left(J-\frac{m}{m-1}+\frac{J^{1-m}}{m-1}\right),$$

$$\mu, k, m = \text{const.},\quad \mu, k > 0,\quad m > \frac{2}{3}, \tag{60}$$

was proposed by Baltz (1969). The condition $m > \frac{2}{3}$ follows from molecular considerations due to Baltz (1969). Clearly, the material (60) satisfies $(10)_{2,3}$ and we have

$$\frac{d^2}{d\lambda^2}[\sqrt{\lambda}W_1(\sqrt{\lambda},\sqrt{\lambda})] = k\lambda^{-m-1}(1-m)+\frac{2\mu}{9\lambda^{4/3}}. \tag{61}$$

The condition (55) is therefore satisfied for $m \leqslant 1$ and thus, for $m \in (\frac{2}{3}, 1]$ and appropriate boundary conditions, the radial deformation of hollow cylindrical tubes composed of this material cannot be accompanied by an overall volume increase. (We note that from the present considerations no conclusions can be drawn about the nature of volume changes that accompany the bending deformations of rectangular blocks composed of this material since the corresponding expression for $(d^2/d\lambda^2)W_1(\sqrt{\lambda},\sqrt{\lambda})$ does not have a definite sign.)

A compressible neo-Hookean material for which the conditions $(10)_{2,3}$ and (37) are satisfied is given by

$$W = (\lambda_1-\lambda_2)^2+\frac{\kappa^2-1}{\kappa}[J(\ln J-1)+1],\quad \kappa = \text{const.},\quad 0<\kappa<1, \tag{62}$$

and a compressible Varga material which satisfies (43) and (44) is given by

$$W = \lambda_1+\lambda_2+\left(\frac{1}{\sqrt{\kappa}}-1\right)(J+1)-\frac{2}{\sqrt{\kappa}}\sqrt{J},\quad \kappa = \text{const.},\quad 0<\kappa<1. \tag{63}$$

We conclude that, for boundary conditions of the type (33) or (34), the bending of rectangular blocks composed of these two materials (both of which have been considered previously by Aron and Wang, 1995) cannot be accompanied by an overall volume increase, and that for boundary conditions which render the left-hand side of (56) non-positive, the radial deformation of hollow cylindrical tubes composed of the material (63) cannot be accompanied by an overall volume increase.

Another example of a material which belongs to the class (27) and which satisfies the conditions (43) and (44) is provided by

$$W = \lambda_1^2\lambda_2^{-1}+3(\lambda_1+\lambda_2)+\lambda_2^2\lambda_1^{-1}-\kappa\sqrt{J}+(\kappa-4)(J+1),\quad \kappa = \text{const.},\quad \kappa>4, \tag{64}$$

and an example of a harmonic material which satisfies $(19)_{2,3}$, (43) and (44) is provided by

$$W = \tfrac{1}{12}(\lambda_1+\lambda_2)^3+\tfrac{1}{4}(\lambda_1+\lambda_2)^2-\lambda_1-\lambda_2+\tfrac{4}{3}-J. \tag{65}$$

For appropriate boundary conditions therefore, (40) will hold for both these materials and in respect to both the bending deformation (30) and the radial deformation (50).

We note that, for some of the materials discussed in this section, solutions that describe the bending of rectangular blocks and/or the radial deformation of cylindrical tubes have been obtained in closed form. For details, we refer the reader to Carroll (1988), Carroll and Horgan (1990), Aron and Wang (1995) and Wang and Aron (1996).

6. DISCUSSION

In this paper it is shown that for certain material classes, non-trivial predictions can be made as to whether the overall volume changes that accompany two types of overall deformation are expansive or contractive and, as discussed in Sections 1 and 3, considerations of this nature may be of assistance to the experimental process in nonlinear elasticity. Each of the material classes under consideration is characterised by a specific set of constitutive assumptions (which are mainly expressed in terms of inequalities) and we emphasise that unless these assumptions hold irrespective of the deformation magnitude, no conclusion may be drawn in the present context (see the discussion regarding the material (60) in Section 5). On the other hand, if on the basis of the present analysis it can be predicted that the volume change in a certain material (such as the 47 vol. % foamed polyurethane rubber which is known to be represented by the Blatz–Ko strain energy function; see Section 5 and Blatz and Ko, 1962) has a definite sign irrespective of the deformation magnitude and if this prediction is subsequently contradicted by experimental findings, it can be concluded that the material model employed in the analysis is not capable of representing the (real) material at all states of deformation. Finally, we remark that, with the exception of inequalities $(2)_1$ and (3), none of the inequalities which have been used in the paper can be expected to be satisfied universally (that is, they cannot be expected to be satisfied by all unconstrained isotropic elastic solids; see the examples discussed in Section 5) and consequently, their use here can be justified only by the fact that they hold for some of the well-known material models which have been employed in the literature.

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